

SPECTRAL MULTIPLICITIES FOR ERGODIC FLOWS

ALEXANDRE I. DANILENKO AND MARIUSZ LEMAŃCZYK

ABSTRACT. Let E be a subset of positive integers such that $E \cap \{1, 2\} \neq \emptyset$. A weakly mixing finite measure preserving flow $T = (T_t)_{t \in \mathbb{R}}$ is constructed such that the set of spectral multiplicities (of the corresponding Koopman unitary representation generated by T) is E . Moreover, for each non-zero $t \in \mathbb{R}$, the set of spectral multiplicities of the transformation T_t is also E . These results are partly extended to actions of some other locally compact second countable Abelian groups.

0. INTRODUCTION

Let G be a locally compact second countable Abelian group and let $T = (T_g)_{g \in G}$ be a measure preserving action of G on a standard probability space (X, \mathfrak{B}, μ) . The spectral theory of dynamical systems studies the corresponding Koopman unitary representation $U_T = (U_T(g))_{g \in G}$ in the Hilbert space $L_0^2(X, \mu) := L^2(X, \mu) \ominus \mathbb{C}$ given by

$$U_T(g)f := f \circ T_{-g}$$

(see [KaT]). Such a representation is completely characterized (up to unitary equivalence) by a measure of maximal spectral type on the dual group \widehat{G} and a spectral multiplicity function $l_T : \widehat{G} \ni w \rightarrow l_T(w) \in \mathbb{N} \cup \{\infty\}$. We denote by $\mathcal{M}(T)$ the (essential) image of l_T .

One of the most appealing open problems in the spectral theory of dynamical systems can be stated as follows: when a unitary representation is unitarily equivalent to a Koopman representation? Let us consider a weak version of this problem by replacing the unitary equivalence with another (weaker) equivalence relation on the set of unitary representations of G . It was introduced in [Fr] for the unitary representations of \mathbb{Z} . Two unitary representations U and V of G in Hilbert spaces \mathcal{H}_U and \mathcal{H}_V respectively are called *cyclicly isomorphic* if there is a unitary operator $W : \mathcal{H}_U \rightarrow \mathcal{H}_V$ such that the image under W of each U -cyclic subspace in \mathcal{H}_U is a V -cyclic subspace in \mathcal{H}_V and vice versa. Based on the proof in [Fr] for $G = \mathbb{Z}$ it is easy to see that if U and V have a continuous spectrum then they are cyclicly isomorphic if and only if $\mathcal{M}(U) = \mathcal{M}(V)$. We thus come to the following natural question which is called the spectral multiplicity problem:

which subsets $E \subset \mathbb{N}$ are realizable as $E = \mathcal{M}(T)$ for an ergodic (or weakly mixing) free action T ?

In the case $G = \mathbb{Z}$ the spectral multiplicity problem was studied by a number of authors (see references in a recent survey [Le] and subsequent progress in [Ry2], [KaL], [Da3]). It is proved, in particular, that a subset $E \subset \mathbb{N}$ is realizable if one of the following is fulfilled: $1 \in E$, $2 \in E$ or $E = n \cdot F$ for some $n \geq 2$ and a subset $F \subset \mathbb{N}$ with $1 \in F$. It is believed that every subset of \mathbb{N} is realizable. In the case $G = \mathbb{Z}^2$, weakly mixing realizations of the subsets $E \ni 1$ were constructed in [Fi] and weakly mixing realizations of subsets $\{2, 4, \dots, 2^n\}$, for each $n > 0$, were shown in [Ko]. For a large class of Abelian locally compact second countable groups G including all countable groups and \mathbb{R}^n , it was proved in [DaS] that there exist weakly mixing G -actions with homogeneous spectrum of arbitrary multiplicity.

In the present paper we mainly consider the case when $G = \mathbb{R}$.

Theorem 0.1 (Main theorem). *Let E be a subset of positive integers such that $E \cap \{1, 2\} \neq \emptyset$.*

- (i) *There is a weakly mixing finite measure preserving flow $T = (T_t)_{t \in \mathbb{R}}$ such that the set of spectral multiplicities of the Koopman unitary representation generated by T is E .*
- (ii) *For each non-zero $t \in \mathbb{R}$, the set of spectral multiplicities of the Koopman operator generated by the transformation T_t is also E .*

Now Theorem 0.1(i) can be interpreted in the following way: every unitary representation of \mathbb{R} with continuous spectrum and such that $\mathcal{M}(U) \cap \{1, 2\} \neq \emptyset$ is cyclicly isomorphic to a Koopman representation of \mathbb{R} . Secondly, given a subset $E \subset \mathbb{N}$ such that $E \cap \{1, 2\} \neq \emptyset$, denote by \mathcal{W}_E the set of weakly mixing transformations S with $\mathcal{M}(S) = E$. As was mentioned above, the set \mathcal{W}_E is known to be non-empty. Theorem 0.1(ii) strengthens this fact: $\mathcal{W}_E \cup \{\text{Id}\}$ contains a one-parameter subgroup.

Now we make some remarks concerning the proof of Theorem 0.1. The simplest way to obtain flows with non-trivial spectral properties is to consider the suspensions of ergodic transformations with non-trivial spectral properties. We recall that the suspensions are special flows constructed under the constant function 1. In other words, they are \mathbb{R} -actions induced by \mathbb{Z} -actions. In Section 1 we briefly review properties of induced actions in a more general setting of pairs (G, H) , where G is a locally compact second countable Abelian group and H a closed co-compact subgroup of G . Some of these properties were established in original papers by G. Mackey [Ma] and R. Zimmer [Zi]. By means of the inducing we can obtain “cheaply” a realization of each subset $E \subset \mathbb{N}$ containing 1 as the set of spectral multiplicities of an ergodic flow. Unfortunately, the condition $1 \in E$ is unavoidable within the class of suspension flows. Moreover, every suspension flow has a non-trivial discrete spectrum. Therefore to construct weakly mixing realizations we apply another approach. It is a continuous analogue of the realizations produced in [Da3]. The desired flows are compact group extensions of either rank-one flows (for realizations of sets $E \ni 1$ in Section 4) or Cartesian squares of rank-one flows (for realizations of sets $E \ni 2$ in Section 5). Sections 2 and 3 contain some preliminary material to understand the techniques used in Sections 4 and 5. In the final Section 6 we partly extend Theorem 0.1 to the actions of other non-compact Abelian groups: connected groups, groups without non-trivial compact subgroups, groups containing a closed one-parameter subgroup, etc.

1. INDUCED ACTIONS

Let G be a locally compact second countable Abelian group and H a co-compact subgroup of G . Given a measure preserving action $S = (S_h)_{h \in H}$ of H on a standard probability space (X, \mathfrak{B}, μ) , we denote by $T = (T_g)_{g \in G}$ the induced action of G on the product space $(G/H \times X, \lambda_{G/H} \times \mu)$, where $\lambda_{G/H}$ is Haar measure on G/H (see [Ma], [Zi]). Fix a Borel cross-section $s : G/H \rightarrow G$ of the natural projection $G \rightarrow G/H$ such that $s(H) = 0$. Then

$$(1-1) \quad T_g(y, x) := (gy, S_{h(g,y)}x),$$

where $h(g, y) := -s(gy) + g + s(y) \in H$. Notice that the mapping $h : G \times Y \rightarrow H$ is a 1-cocycle, i.e.

$$h(g_1 g_2, y) = h(g_1, g_2 y) + h(g_2, y)$$

for all $y \in Y$, $g_1, g_2 \in G$. If S is ergodic then so is T . Denote by U_T and U_S the Koopman representations of G and H generated by T and S respectively. Then U_T is unitarily equivalent to the unitary representation of G induced by U_S [Ma]. Recall that given a unitary representation $V = (V_h)_{h \in H}$ of H in a Hilbert space \mathcal{H} , the *induced (by V) representation* $U = (U_g)_{g \in G}$ of G is defined on a Hilbert space $L^2(G/H, \lambda_{G/H}) \otimes \mathcal{H}$ by the formula

$$U_{-g}f(y) := V_{h(g,y)}(f(gy)).$$

Here we consider $f \in L^2(G/H, \lambda_{G/H}) \otimes \mathcal{H}$ as a measurable function $f : G/H \rightarrow \mathcal{H}$ such that $\int_{G/H} \|f(y)\|^2 d\lambda_{G/H}(y) < \infty$. In particular, under the above identification, if $b \in L^2(G/H, \lambda_{G/H})$ and $a \in \mathcal{H}$ then for $f(y) = (b \otimes a)(y) = b(y)a$ and $h \in H$ we obtain

$$(U_h f)(y) = b(y)V_h(a) = (b \otimes V_h a)(y).$$

Proposition 1.1. *Let $\pi : \widehat{G} \rightarrow \widehat{H}$ stand for the natural projection. Denote by σ_U a measure of maximal spectral type of U on \widehat{G} .*

- (i) $\sigma_U \circ \pi^{-1}$ is a measure of maximal spectral type of V .
- (ii) U and V have the same set of spectral multiplicities.

Proof. Let $M \subset \mathbb{N} \cup \{\infty\}$ denote the set of spectral multiplicities of V . Then there is a decomposition $\mathcal{H} = \bigoplus_{i \in M} \bigoplus_{j=1}^i \mathcal{H}_{i,j}$ of \mathcal{H} such that

- $\mathcal{H}_{i,j}$ is a cyclic space for V for every pair (i, j) . Denote by $\sigma_{i,j}$ a measure of the maximal spectral type for $V \upharpoonright \mathcal{H}_{i,j}$. Then
- $\sigma_{i,j} \perp \sigma_{i',j'}$ if $i \neq i'$ and
- $\sigma_{i,j} \sim \sigma_{i',j'}$ if $i = i'$.

It is easy to see that $L^2(G/H, \lambda_{G/H}) \otimes \mathcal{H}_{i,j}$ is a cyclic space for U . Denote by $\sigma'_{i,j}$ a measure of the maximal spectral type of $U \upharpoonright (L^2(G/H, \lambda_{G/H}) \otimes \mathcal{H}_{i,j})$. It is easy to see that

$$(1-2) \quad \sigma'_{i,j} \sim \sigma'_{i',j'} \quad \text{if} \quad i = i'.$$

Let $a \in \mathcal{H}$ be a cyclic vector for V such that the spectral measure of a is σ_V . Take a unit vector $b \in L^2(G/H, \lambda_{G/H})$. Then for each $h \in H$,

$$\langle U_h(b \otimes a), b \otimes a \rangle = \langle V_h a, a \rangle$$

This implies that π projects the spectral measure of $b \otimes a$ into σ_V . This yields $\sigma'_{i,j} \circ \pi^{-1} = \sigma_{i,j}$ for each pair (i, j) . Therefore

$$(1-3) \quad \sigma'_{i,j} \perp \sigma'_{i',j'} \quad \text{if } i \neq i'.$$

Since $L^2(G/H, \lambda_{G/H}) \otimes \mathcal{H} = \bigoplus_{i \in M} \bigoplus_{j=1}^i L^2(G/H, \lambda_{G/H}) \otimes \mathcal{H}_{i,j}$, we deduce both (i) and (ii) from (1-2) plus (1-3). \square

It follows that if a G -action T is induced by an H -action S then $\mathcal{M}(T) = \mathcal{M}(S) \cup \{1\}$. The “extra” value 1 appears because $L_0^2(G/H, \lambda_{G/H}) \otimes 1$ is a U_T -cyclic subspace of $L_0^2(G/H \times X, \lambda_{G/H} \times \mu)$.

Denote by V an action of G on the homogeneous space G/H by translations. The following proposition about induced actions was shown by R. Zimmer in [Zi].

Proposition 1.2.

- (i) *Let T be an action of G on (X, \mathfrak{B}, μ) . Then the action of G induced by $T \upharpoonright H$ is isomorphic to the Cartesian product $V \times T$.*
- (ii) *An action T of G is induced by an action of H if and only if T has a factor isomorphic to V .*

Recall that given a dynamical system (Z, ν, T) , a $T \times T$ -invariant measure ρ on the product space $Z \times Z$ such that the coordinate marginals of ρ are both equal to ν is called a *(2-fold) self-joining of T* . For the theory of joinings and notions like relative weak mixing, relative compactness, simplicity and centralizer we refer the reader to [JuR] and [KaT].

In the following corollary we describe the structure of self-joinings of induced actions.

Proposition 1.3. *Let T be a G -action induced by an ergodic H -action S (see (1-1) for the notation). Let ρ be an ergodic self-joining of T . Then $(Y \times X \times Y \times X, \rho, T \times T)$ is an induced G -action. More precisely, there are $\kappa \in J_2^e(S)$ and $g \in G$ such that*

$$\rho = \int_{G/H} \kappa \circ (S_{h(s(y), y)} \times S_{h(s(y), gy)}) \times \delta_y \times \delta_{gy} d\lambda_{G/H}(y).$$

ρ is a graph of an isomorphism if and only if so is κ . Hence two induced G -actions are isomorphic if and only if the underlying H -actions are isomorphic.

Proof. We use the notation from (1.1). The projection map $Y \times X \rightarrow Y$ intertwines T with V (see (1.1)). Therefore the projection ρ^* of ρ to $Y \times Y$ is an ergodic self-joining of V . Hence there is $g \in G$ such that $\rho^*(A \times B) = \lambda_{G/H}(A \times gB)$ for all measurable subsets $A, B \subset G/H$. Disintegrate now ρ with respect to ρ^* :

$$(1-4) \quad \rho = \int_{G/H} \kappa_y \times \delta_y \times \delta_{gy} d\lambda_{G/H}(y),$$

where $Y \times Y \ni (y, y) \mapsto \kappa_y$ is a measurable field of probability measures on $X \times X$ such that

$$(1-5) \quad \int_Y \kappa_y^{(1)} \times \delta_y d\lambda_{G/H}(y) = \int_Y \kappa_y^{(2)} \times \delta_{gy} d\lambda_{G/H}(y) = \mu \times \lambda_{G/H},$$

where $\kappa_y^{(i)}$ is the i -th coordinate projection of κ_y for $i = 1, 2$ and every $y \in Y$. Since ρ is $T \times T$ -invariant, we deduce from (1-4) and (1-1) that

$$(1-6) \quad \kappa_{g'y} = \kappa_y \circ (S_{h(g',y)} \times S_{h(g',gy)})$$

for each $g' \in G$ at a.e. $y \in G/H$. Substituting $g' \in H$ into (1-6) we obtain that κ_y is invariant under $S \times S$ for a.a. $y \in G/H$. Since μ is ergodic under S , we deduce from (1-5) that $\kappa_y^{(1)} = \kappa_y^{(2)} = \mu$ for a.a. $y \in G/H$. Thus κ_y is a self-joining of S for a.a. $y \in G/H$. Since G acts transitively on Y , the equation (1-6) can be “resolved” in a standard way:

$$\kappa_y = \kappa \circ (S_{h(s(y),y)} \times S_{h(s(y),gy)}), \quad y \in G/H,$$

for certain self-joining κ of S (formally, put $\kappa = \kappa_H$ and $g' = s(y)$ into (1-6)). Moreover, κ is ergodic.

The remaining assertions of Proposition 1.3 follow immediately. \square

Corollary 1.4.

- (i) *If S is either relatively weakly mixing or relatively compact with respect to some factor \mathfrak{A} then T is either relatively weakly mixing or relatively compact (respectively) with respect to the factor induced by \mathfrak{A} .*
- (ii) *T is simple if and only if S has pure point spectrum.*
- (iii) $C(T) = \{(\text{Id} \times R)T_g \mid g \in G, R \in C(S)\}$.
- (iv) *If \mathfrak{F} is a factor of T that contains the standard factor V then \mathfrak{F} is an induced action of a factor of S .*

We note that T may also have factors which do not contain V (for instance in the case considered in Proposition 1.2(i)).

2. PRELIMINARIES

We start with an important algebraic lemma. Let G be a countable Abelian group, H a subgroup of G and $v : G \rightarrow G$ a group automorphism. We set

$$L(G, H, v) := \{\#(\{v^i(h) \mid i \in \mathbb{Z}\} \cap H), h \in H \setminus \{0\}\}.$$

Algebraic Lemma 2.1 ([KwL], [Da3]). *Given any subset $E \subset \mathbb{N}$, there exist a countable Abelian group G , a subgroup $H \subset G$ and an automorphism $v : G \rightarrow G$ such that*

- (i) $E = L(G, H, v)$ and
- (ii) *the subgroup of \widehat{v} -periodic points in \widehat{G} is countable and dense.*

We now recall the definition of rank one. Let $S = (S_g)_{g \in \mathbb{R}^d}$ be a measure preserving action of \mathbb{R}^d on a standard σ -finite measure space (Y, \mathfrak{C}, ν) .

Definition 2.2.

- (i) A *Rokhlin tower or column* for S is a triple (A, f, F) , where $A \in \mathfrak{C}$, F is a cube in \mathbb{R}^d and $f : A \rightarrow F$ is a measurable mapping such that for any Borel subset $H \subset F$ and an element $g \in \mathbb{R}^d$ with $g + H \subset F$, one has $f^{-1}(g + H) = S_g f^{-1}(H)$.

- (ii) S is said to be of *rank one (by cubes)* if there exists a sequence of Rokhlin towers (A_n, f_n, F_n) such that the volume of F_n goes to infinity and for any subset $C \in \mathfrak{C}$ of finite measure, there is a sequence of Borel subsets $H_n \subset F_n$ such that

$$\lim_{n \rightarrow \infty} \nu(C \triangle f_n^{-1}(H_n)) = 0.$$

Denote by $\mathcal{R} \subset X \times X$ the T -orbit equivalence relation. A Borel map α from \mathcal{R} to a compact group K is called a *cocycle* of \mathcal{R} if

$$\alpha(x, y) + \alpha(y, z) = \alpha(x, z) \quad \text{for all } (x, y), (y, z) \in \mathcal{R}.$$

Two cocycles $\alpha, \beta : \mathcal{R} \rightarrow K$ are *cohomologous* if there is a μ -conull subset $B \subset X$ such that

$$\alpha(x, y) = \phi(x) + \beta(x, y) - \phi(y) \quad \text{for all } (x, y) \in \mathcal{R} \cap (B \times B).$$

for a Borel map $\phi : X \rightarrow K$. Given a cocycle $\alpha : \mathcal{R} \rightarrow K$ and a closed subgroup $H \subset K$, we can define a new flow $T^{\alpha, H} = (T_t^{\alpha, H})_{t \in \mathbb{R}}$ on the space $(X \times K/H, \mu \times \lambda_{K/H})$ by setting

$$T_t^{\alpha, H}(x, k + H) = (T_t x, \alpha(T_t x, x) + k + H).$$

This flow is called a *compact group extension* of T . Given a character $\chi \in \widehat{K}$, we denote by $U_{T^{\alpha, \chi}}$ the following unitary representation of \mathbb{R} in $L^2(X, \mu)$:

$$(U_{T^{\alpha, \chi}}(t)f)(x) := \chi(\alpha(T_{-t}x, x))f(T_{-t}x).$$

There is a natural decomposition of $U_{T^{\alpha, H}}$ into an orthogonal sum

$$U_{T^{\alpha, H}} = \bigoplus_{\chi \in \widehat{K/H}} U_{T^{\alpha, \chi}},$$

where $\widehat{K/H}$ is considered as a subgroup of \widehat{K} .

If a transformation S commutes with T (i.e. $S \in C(T)$) then a cocycle $\alpha \circ S : \mathcal{R} \rightarrow K$ is well defined by $\alpha \circ S(x, y) := \alpha(Sx, Sy)$. The important cohomology equation on α mentioned in Section 0 can now be stated as follows

$$(2-1) \quad \alpha \circ S \text{ is cohomologous to } v \circ \alpha$$

for some $S \in C(T)$ and a group automorphism $v : K \rightarrow K$.

3. (C, F) -FLOWS AND (C, F) -COCYCLES

To prove Main Theorem we will use the (C, F) -construction (see [Da1] and references therein). We now briefly outline its formalism. Let two sequences $(C_n)_{n \geq 0}$ and $(F_n)_{n \geq 0}$ of subsets in \mathbb{R} be given such that:

- $F_n = [0, h_n)$, $h_0 = 1$,
- C_n is finite, $\#C_n > 1$, $\min C_n = 0$,
- $F_n + C_{n+1} \subset [0, h_{n+1} - 1)$,
- $(F_n + c) \cap (F_n + c') = \emptyset$ if $c \neq c'$, $c, c' \in C_{n+1}$,
- $\lim_{n \rightarrow \infty} \frac{h_n}{\#C_1 \cdots \#C_n} < \infty$.

Let $X_n := F_n \times C_{n+1} \times C_{n+2} \times \cdots$. Endow this set with the standard product Borel structure. The following map

$$(f_n, c_{n+1}, c_{n+2}) \mapsto (f_n + c_{n+1}, c_{n+2}, \dots)$$

is a Borel embedding of X_n into X_{n+1} . We now set $X := \bigcup_{n \geq 0} X_n$ and endow it with the inductive limit standard Borel structure. Given a Borel subset $A \subset F_n$, we denote by $[A]_n$ the following cylinder: $\{x = (f, c_{n+1}, \dots) \in X_n \mid f \in A\}$. The family of all cylinders generates the entire σ -algebra \mathfrak{B} on X .

Let \mathcal{R} stand for the *tail* equivalence relation on X : two points $x, x' \in X$ are \mathcal{R} -equivalent if there is $n > 0$ such that $x = (f_n, c_{n+1}, \dots)$, $x' = (f'_n, c'_{n+1}, \dots) \in X_n$ and $c_m = c'_m$ for all $m > n$. Of course, \mathcal{R} is a Borel subset of $X \times X$. It is easy to see that there is only one probability (non-atomic) Borel measure μ on X which is invariant under \mathcal{R} . This means that every Borel isomorphism of X whose graph is a subset of \mathcal{R} preserves μ . We note that the restriction of μ on X_n is an infinite product $\nu_n \times \kappa_{n+1} \times \kappa_{n+2} \times \cdots$, where κ_i is the equidistribution on C_i and ν_{n+1} is a measure proportional to $\lambda_{\mathbb{R}} \upharpoonright F_n$. Hence for each $n \geq 0$ and a subset $A \subset F_n$,

$$\mu([A]_n)/\mu(X_n) = \lambda_{\mathbb{R}}(A)/h_n.$$

We now isolate a subset $\tilde{X} \subset X$ such that

$$\tilde{X} \cap X_n := \{x = (f_n, c_{n+1}, c_{n+2}, \dots) \in X_n \mid c_k \neq 0 \text{ infinitely often}\}.$$

Then X_n is Borel, \mathcal{R} -saturated and $\mu(\tilde{X}) = 1$. Now we define a Borel flow $T = (T_t)_{t \in \mathbb{R}}$ on \tilde{X} by setting

$$T_t(f_n, c_{n+1}, \dots) := (t + f_n, c_{n+1}, \dots) \text{ whenever } t + f_n < h_n, \ n > 0.$$

This formula defines T_t partly on \tilde{X} . When $n \rightarrow \infty$, T_t extends to the entire \tilde{X} . It is easy to see that the mapping $\tilde{X} \times \mathbb{R} \ni (x, t) \mapsto T_t x \in \tilde{X}$ is Borel and $T_{t_1} T_{t_2} = T_{t_1 + t_2}$ for all $t_1, t_2 \in \mathbb{R}$. Moreover, the T -orbit equivalence relation coincides with $\mathcal{R} \upharpoonright \tilde{X}$. It follows that T is μ -preserving. In what follows we do not distinguish objects (sets, transformations, etc.) if they agree a.e. That is why we consider that T is defined on the entire X .

Definition 3.1. We call T the (C, F) -flow associated with $(C_{n+1}, F_n)_{n \geq 0}$.

It is easy to see that T is of rank one. Hence it is free and ergodic.

We recall a concept of (C, F) -cocycle (see [Da2]). From now on, the group K is assumed Abelian. Given a sequence of maps $\alpha_n : C_n \rightarrow K$, $n = 1, 2, \dots$, we first define a Borel cocycle $\alpha : \mathcal{R} \cap (X_0 \times X_0) \rightarrow K$ by setting

$$\alpha(x, x') := \sum_{n > 0} (\alpha_n(c_n) - \alpha_n(c'_n)),$$

whenever $x = (0, c_1, c_2, \dots) \in X_0$, $x' = (0, c'_1, c'_2, \dots) \in X_0$ and $(x, x') \in \mathcal{R}$. To extend α to the entire \mathcal{R} , we first define a map $\pi : X \rightarrow X_0$ as follows. Given $x \in X$, let n be the least positive integer such that $x \in X_n$. Then $x = (f_n, c_{n+1}, \dots) \in X_n$. We set

$$\pi(x) := (\underbrace{0, \dots, 0}_{n+1 \text{ times}}, c_{n+1}, c_{n+2}, \dots) \in X_0.$$

Of course, $(x, \pi(x)) \in \mathcal{R}$. Now for each pair $(x, y) \in \mathcal{R}$, we let

$$\alpha(x, y) := \alpha(\pi(x), \pi(y)).$$

It is easy to verify that α is a well defined cocycle of \mathcal{R} with values in K .

Definition 3.2. We call α the (C, F) -cocycle associated with $(\alpha_n)_{n=1}^\infty$.

Suppose we have an invertible measure preserving transformation S of (X, μ) such that S maps bijectively $\mathcal{R}(x)$ on $\mathcal{R}(Sx)$ for μ -a.a. $x \in X$. (This condition holds, for instance, if $S \in C(T)$.) Then for each cocycle $\alpha : \mathcal{R} \rightarrow K$, we can define a cocycle $\alpha \circ S$ by setting

$$\alpha \circ S(x, y) := \alpha(Sx, Sy), \quad (x, y) \in \mathcal{R}.$$

Adapting the argument from [Da2, Section 4] we obtain the following lemma.

Lemma 3.3. Let $\bar{z} = (z_n)_{n=1}^\infty$ be a sequence of positive reals. Suppose that

$$\sum_{n>0} \#(C_n \triangle (C_n - z_n)) / \#C_n < \infty.$$

For each $m > 0$, we set

$$X_m^{\bar{z}} := [0, h_m - z_1 - \cdots - z_m) \times \prod_{n>m} (C_n \cap (C_n - z_n)) \subset X_m.$$

Then a transformation $S_{\bar{z}}$ of (X, μ) is well defined by setting

$$(3-1) \quad S_{\bar{z}}(x) := (z_1 + \cdots + z_m + f_m, z_{m+1} + c_{m+1}, z_{m+2} + c_{m+2}, \dots)$$

for all $x = (f_m, c_{m+1}, c_{m+2}, \dots) \in X_m^{\bar{z}}$, $m = 1, 2, \dots$. Moreover, $S_{\bar{z}}$ commutes with T and $T^{z_1 + \cdots + z_m} \rightarrow S_{\bar{z}}$ weakly as $m \rightarrow \infty$.

Let v be a continuous group automorphism of K and let

$$C_m^\circ := \{c \in C_m \cap (C_m - z_m) \mid \alpha_m(c + z_m) = v(\alpha_m(c))\}.$$

If

$$(3-2) \quad \sum_{n>0} (1 - \#C_n^\circ / \#C_n) < \infty$$

then the cocycle $\alpha \circ S_{\bar{z}}$ is cohomologous to $v \circ \alpha$.

4. REALIZATION OF SETS CONTAINING 1 AS SPECTRAL MULTIPLICITIES

Let E be a subset of positive integers. By Algebraic Lemma 2.1, there exist a compact Polish Abelian group K , a closed subgroup H of K and a continuous automorphism v of K such that

$$E = L(\widehat{K}, \widehat{K/H}, \widehat{v}).$$

The subgroup of v -periodic points in K will be denoted by \mathcal{K} . It is countable and dense in K by Lemma 2.1. Let ξ_1 and ξ_2 be two rationally independent positive reals in \mathbb{R} . Fix a partition

$$\mathbb{N} = \mathcal{W}_1 \sqcup \mathcal{W}_2 \sqcup \bigsqcup_{a \in \mathcal{K}} \mathcal{N}_a$$

of \mathbb{N} into infinite subsets. To construct the desired realization we define inductively a sequence $(C_n, h_n, \alpha_n)_{n=1}^\infty$, where C_n a finite subset of \mathbb{R} , h_n is a positive real and $\alpha_n : C_n \rightarrow \mathbb{R}$ a mapping. Suppose we have already constructed this sequence up to index n . Consider two cases.

If $n+1 \in \mathcal{N}_a$ for some $a \in \mathcal{K}$ then we denote by m_a the least positive period of a under v . Now we set

$$\begin{aligned} z_{n+1} &:= m_a n h_n, \quad r_n := n^3 m_a, \\ C_{n+1} &:= h_n \cdot \{0, 1, \dots, r_n - 1\}, \\ h_{n+1} &:= r_n h_n + 1, \end{aligned}$$

Let $\alpha_{n+1} : C_{n+1} \rightarrow K$ be any map satisfying the following conditions

- (A1) $\alpha_{n+1}(c + z_{n+1}) = v \circ \alpha_{n+1}(c)$ for all $c \in C_{n+1} \cap (C_{n+1} - z_{n+1})$,
- (A2) for each $0 \leq i < m_a$ there is a subset $C_{n+1,i} \subset C_{n+1}$ such that

$$\begin{aligned} C_{n+1,i} - h_n &\subset C_{n+1}, \\ \alpha_{n+1}(c) &= \alpha_{n+1}(c - h_n) + v^i(a) \text{ for all } c \in C_{n+1,i} \text{ and} \\ \left| \frac{\#C_{n+1,i}}{\#C_{n+1}} - \frac{1}{m_a} \right| &< \frac{2}{nm_a}. \end{aligned}$$

If $n+1 \in \mathcal{W}_i$ for $i = 1, 2$ then we set

$$\begin{aligned} C_{n+1} &:= \{j h_n \mid 0 \leq j < n\} \sqcup \{j(h_n + \xi_i) + n h_n \mid 0 \leq j < n\}, \\ h_{n+1} &:= 2n h_n + n \xi_i, \\ \alpha_{n+1}(c) &:= 1_K \quad \text{for all } c \in C_{n+1}. \end{aligned}$$

Thus, $C_{n+1}, h_{n+1}, \alpha_{n+1}$ are completely defined.

Denote by (X, μ, T) the (C, F) -flow associated with the sequence $(C_{n+1}, F_n)_{n \geq 0}$, where $F_n := [0, h_n)$. Let \mathcal{R} stand for the tail equivalence relation (or, equivalently, the T -orbit equivalence relation) on X . Denote by $\alpha : \mathcal{R} \rightarrow K$ the cocycle of \mathcal{R} associated with the sequence $(\alpha_n)_{n \geq 0}$. Let $\lambda_{K/H}$ stand for the Haar measure on K/H . We denote by $T^{\alpha, H}$ the following flow on the space $(X \times K/H, \lambda_{K/H})$:

$$T_t^{\alpha, H}(x, k + H) := (T_t x, \alpha(T_t x, x) + k + H), \quad t \in \mathbb{R}.$$

Our purpose in this section is to prove the following theorem.

Theorem 4.1. $\mathcal{M}(T^{\alpha, H}) = E \cup \{1\}$.

Since

$$\sum_{n > 0} \frac{\#(C_n \triangle (C_n - z_n))}{\#C_n} = \sum_{n > 0} \frac{2}{n^2},$$

it follows from Lemma 3.3 that a transformation $S_{\bar{z}}$ of (X, μ) is well defined by the formula (3-1) and $S_{\bar{z}} \in C(T)$.

It follows from (A1) and (A3) that (3-2) is satisfied. Hence by Lemma 3.3,

(4-1) the cocycle $\alpha \circ S_{\bar{z}}$ is cohomologous to $v \circ \alpha$.

We need more notation. Given $a \in \mathcal{K}$ and $\chi \in \widehat{K}$, let

$$l_\chi(a) := m_a^{-1} \sum_{i=0}^{m_a-1} \chi(v^i(a)),$$

where m_a stands for the least positive period of a under v .

Lemma 4.2. *Let $\chi \in \widehat{K}$. Then*

- (i) $U_{T^\alpha, \chi}(h_n) \rightarrow l_\chi(a) \cdot I$ as $\mathcal{N}_a - 1 \ni n \rightarrow \infty$, $a \in \mathcal{K}$.
- (ii) $U_{T^\alpha, \chi}(h_n) \rightarrow 0.5(I + U_{T^\alpha, \chi}(-\xi_j))$ as $\mathcal{W}_j - 1 \ni n \rightarrow \infty$, $j = 1, 2$.

Proof. We will show only (i). The other claim is shown in a similar way. Let $n \in \mathcal{N}_a$. Take any subset $A \subset F_n$. We note that $[A]_n = [A + C_{n+1}]_{n+1}$. Therefore it follows from (A4) that

$$\begin{aligned} U_{T^\alpha, \chi}(h_n)1_{[A]_n}(x) &= \sum_{i=0}^{m_a-1} \chi(\alpha(x, T_{-h_n}x))1_{[A+C_{n+1}, i]_{n+1}}(T_{-h_n}x) + \bar{o}(x) \\ &= \sum_{i=0}^{m_a-1} \chi(v^i(a))1_{[A+C_{n+1}, i+h_n]_{n+1}}(x) + \bar{o}(x) \end{aligned}$$

where $\bar{o}(x)$ is a function whose L^2 -norm is small. Hence

$$U_{T^\alpha, \chi}(h_n) - \sum_{i=0}^{m_a-1} \chi(v^i(a))1_{[C_{n+1}, i+h_n]_{n+1}} \rightarrow 0$$

weakly as $\mathcal{N}_a - 1 \ni n \rightarrow \infty$, where the function $1_{[C_{n+1}, i+h_n]_{n+1}} \in L^\infty(X, \mu)$ is considered as a multiplication operator in $L^2(X, \mu)$.

It remains to use the inequalities from (A2) and a standard fact that for any sequence $C'_n \subset C_n$ such that $\#C'_n/\#C_n \rightarrow \delta$ for some $\delta > 0$ we have

$$1_{[C'_n]_n} \rightarrow \delta I \quad \text{weakly as } n \rightarrow \infty.$$

□

Proof of Theorem 4.1. We first verify that T^α is weakly mixing. Let $U_{T^\alpha}(t)f = \exp(i\lambda t)f$ for some $f \in L^2(X \times K)$, $f \neq 0$ and $\lambda \in \mathbb{R}$. It follows from Lemma 4.2(ii) that

$$U_{T^\alpha}(h_n) \rightarrow 0.5(I + U_{T^\alpha}(-\xi_j))$$

and hence

$$\exp(ih_n\lambda) \rightarrow 0.5(1 + \exp(-i\lambda\xi_j))$$

as $\mathcal{W}_j - 1 \ni n \rightarrow \infty$, $j = 1, 2$. Therefore $|1 + \exp(-i\lambda\xi_j)| = 2$ which implies $\exp(-i\lambda\xi_j) = 1$ for $j = 1, 2$. Since ξ_1 and ξ_2 are rationally independent, $\lambda = 0$. It remains to show that T^α is ergodic. If $\chi \neq 1$ then there is $a \in \mathcal{K}$ with $l_\chi(a) \neq 1$. If $f \in L^2(X, \mu)$ is invariant under $U_{T^\alpha, \chi}$ then Lemma 4.2(i) yields $f = l_\chi(a)f$. Hence $f = 0$. If $\chi = 1$ then $U_{T^\alpha, \chi} = U_T$. Since T is ergodic, each $U_{T^\alpha, \chi}$ -invariant function is constant. Thus, we have shown that U_{T^α} is weakly mixing. Hence $U_{T^\alpha, H}$ is also weakly mixing.

To show that $\mathcal{M}(T^{\alpha, H}) = E \cup \{1\}$ we consider a natural decomposition of $U_{T^\alpha, H}$ into an orthogonal sum

$$U_{T^\alpha, H} = \bigoplus_{\chi \in \widehat{K/H}} U_{T^\alpha, \chi}.$$

It is enough to prove the following:

- (a) $U_{T^\alpha, \chi}$ has a simple spectrum for each χ ,

- (b) $U_{T^\alpha, \chi}$ and $U_{T^\alpha, \xi}$ are unitarily equivalent if χ and ξ belong to the same \widehat{v} -orbit,
- (c) the measures of maximal spectral type of $U_{T, \chi}$ and $U_{T, \xi}$ are mutually singular if χ and ξ belong to different \widehat{v} -orbits.

For each $\epsilon > 0$ and $n > 0$, there are a partition of F_n into intervals $\Delta_0, \dots, \Delta_{M_n}$ and reals t_1, \dots, t_{M_n} such that $\max_j \text{diam } \Delta_j < \epsilon$, $\Delta_j = T_{t_j} \Delta_0$ and the mapping $[\Delta_j]_n \ni x \mapsto \alpha(T_{-t_j} x, x) \in K$ is constant for each $1 \leq j \leq M_n$. This implies (a).

It is straightforward that (4-1) implies (b).

If χ and η are non-equivalent then there is $a \in \mathcal{K}$ such that $l_\chi(a) \neq l_\eta(a)$. Moreover, $U_{T, \chi}^{h_n} \rightarrow l_\chi(a)I$ and $U_{T, \eta}^{h_n} \rightarrow l_\eta(a)I$ as $\mathcal{N}_a - 1 \ni n \rightarrow \infty$ by Lemma 4.2(i). Hence the measures of maximal spectral types of $U_{T, \eta}$ and $U_{T, \chi}$ are mutually singular. Thus (c) holds. \square

Now we are going to show the following claim.

Proposition 4.3. $\mathcal{M}(T_t) = E$ for each $t \neq 0$.

For that we need an auxiliary statement from [LeP]. Given a Borel measure σ on \mathbb{R} , we let $A_\sigma := \{t \in \mathbb{R} \mid \sigma * \delta_t \not\leq \sigma\}$.

Lemma 4.4([LeP]). *Let σ be a finite Borel measure on \mathbb{R} . If there are an analytic function a on \mathbb{R} and a sequence of continuous characters $\xi_n \in \widehat{\mathbb{R}}$ such that $\xi_n \rightarrow \infty$ in $\widehat{\mathbb{R}}$ and $\xi_n \rightarrow a$ weakly in $L^2(\mathbb{R}, \sigma)$ then for each $t_0 \in A_\sigma$ there exists $c \in \mathbb{C}$ with $|c| = 1$ and $a(t + t_0) = ca(t)$ for each $t \in \mathbb{R}$.*

Proof of Proposition 4.3. Denote by σ_T a probability measure of maximal spectral type for T . We first show that $A_{\sigma_T} = \{0\}$. It follows from Lemma 4.2(ii) that

$$U_{T^{\alpha, H}}(h_n) \rightarrow 0.5(I + U_{T^{\alpha, H}}(-\xi_j))$$

weakly as $\mathcal{W}_j - 1 \ni n \rightarrow \infty$, $j = 1, 2$. We deduce from this and Lemma 4.4 that for each $t_0 \in A_{\sigma_T}$ and $j = 1, 2$, there exists a complex number c_j such that

$$1 + \exp(2\pi i \xi_j(t + t_0)) = c_j(1 + \exp(2\pi i \xi_j t))$$

for all $t \in \mathbb{R}$. This yields $c_j = 1$ and $\exp(2\pi i \xi_j t_0) = 1$ for $j = 1, 2$. Since ξ_1 and ξ_2 are rationally independent, $t_0 = 0$.

Thus if $0 \neq t \in \mathbb{R}$ then $\sigma_T * \delta_t \perp \sigma_T$. Hence the natural projection $\mathbb{R} \rightarrow \mathbb{R}/t\mathbb{Z}$ is one-to-one on a subset of full σ_T -measure. This implies that $\mathcal{M}(T^{\alpha, H}) = \mathcal{M}(T_t^{\alpha, H})$. \square

5. REALIZATION OF SETS CONTAINING 2 AS SPECTRAL MULTIPLICITIES

Now let E be a subset of \mathbb{N} such that $1 \notin E$. In this section we will realize the set $E \cup \{2\}$. We first prove a couple of auxiliary lemmata.

Lemma 5.1. *Let T be a weakly mixing flow with a simple spectrum. Let ξ_1, ξ_2 be two rationally independent reals. Suppose that the weak closure $WC(U_T)$ of the group $\{U_T(t) \mid t \in \mathbb{R}\}$ contains the following operators:*

$$(5-1) \quad 0.5(I + U_T(j\xi_1)), \quad 0.5(I + U_T(\xi_2)) \quad \text{and} \quad 0.5(I + U_T(\xi_2 - \xi_1)),$$

$j = 1, 2$. Then the product flow $T \times T := (T_t \times T_t)_{t \in \mathbb{R}}$ has a homogeneous spectrum of multiplicity 2 in the orthocomplement to the constants.

Proof. Let h be a cyclic vector for U_T . Denote by \mathcal{C} the closure of the span of 3 vectors $h \otimes h, U_T(\xi_1)h \otimes h$ and $U_T(\xi_2)h \otimes h$. It follows from (5-1) that \mathcal{C} is invariant under the following operators:

$$(5-2) \quad U_T(j\xi_1) \otimes I + I \otimes U_T(j\xi_1), \quad j = 1, 2,$$

$$(5-3) \quad U_T(\xi_2) \otimes I + I \otimes U_T(\xi_2),$$

$$(5-4) \quad U_T(\xi_2 - \xi_1) \otimes I + I \otimes U_T(\xi_2 - \xi_1).$$

Slightly modifying the argument from [Ag] and [Ry1], we deduce from (5-2) and (5-3) that

$$U_T(n\xi_1)h \otimes h \in \mathcal{C} \quad \text{and} \quad U_T(n\xi_2)h \otimes h \in \mathcal{C}$$

for all $n \in \mathbb{Z}$. Applying (5-4) to $U(\xi_1)h \otimes h$ we obtain that $U_T(2\xi_1 - \xi_2)h \otimes h \in \mathcal{C}$. Applying (5-2) with $j = -2$ step by step infinitely many times and then with $j = 2$ infinitely many times we obtain that $U(2n\xi_1 - \xi_2)h \otimes h \in \mathcal{C}$ for each $n \in \mathbb{Z}$. Next, applying (5-3) to $U(2\xi_1 - \xi_2)h \otimes h$, we deduce that $U(2\xi_1 - 2\xi_2) \in \mathcal{C}$. Then again apply infinitely many times (5-2) with $j = -2$ and $j = 2$ to obtain $U(2n\xi_1 - 2\xi_2)h \otimes h \in \mathcal{C}$. And so on. Finally, we obtain that

$$U(2n\xi_1 + m\xi_2)h \otimes h \in \mathcal{C} \quad \text{for all } n, m \in \mathbb{Z}.$$

Hence $U(t)h \otimes h \in \mathcal{C}$ for all $t \in \mathbb{R}$. Since h is cyclic for U , it follows that $\mathcal{H} \otimes h \subset \mathcal{C}$ and therefore $\mathcal{C} = \mathcal{H} \otimes \mathcal{H}$.

Denote by m the spectral multiplicity function for $(T_t \times T_t)_{t \in \mathbb{R}}$ and denote by σ the measure of maximal spectral type for T . By the above, $m(\lambda) \leq 3$ for $\sigma * \sigma$ -a.a. $\lambda \in \mathbb{R}$.

On the other hand, since T is weakly mixing and $\sigma \times \sigma = \int_{\mathbb{R}} \sigma_\lambda d\sigma * \sigma(\lambda)$ and σ_λ is invariant under the flip mapping $\mathbb{R}^2 \ni (y, z) \mapsto (z, y) \in \mathbb{R}^2$, it follows that $m(\lambda) \in \{2, 4, \dots\} \cup \{\infty\}$. Hence $m(\lambda) = 2$ a.e. \square

Lemma 5.2. *Let U and V be unitary representations of \mathbb{R} with simple spectrum. Assume that there are sequences $a_n \rightarrow \infty, b_n \rightarrow \infty, a'_n \rightarrow \infty$ and $b'_n \rightarrow \infty$ such that*

- (i) $U(a_n) \rightarrow 0.5(I + U(\xi))$ and $V(a_n) \rightarrow 0.5(I + V(\xi))$,
- (ii) $U(b_n) \rightarrow 0.5(dI + U(\xi))$ and $V(b_n) \rightarrow 0.5(eI + V(\xi))$,
- (iii) $U(a'_n) \rightarrow 0.5(I + U(\eta))$ and $V(a'_n) \rightarrow 0.5(I + V(\eta))$ and
- (iv) $U(b'_n) \rightarrow 0.5(d'I + U(\eta))$ and $V(b'_n) \rightarrow 0.5(e'I + V(\eta))$

for some $\xi, \eta, d, e, d', e' \in \mathbb{R}$. If $d \neq e, d' \neq e'$ and ξ and η are rationally independent then $U \otimes V$ has also a simple spectrum.

Proof. Let v_1 and v_2 be cyclic vectors for U and V . Denote by \mathcal{C} the $U \otimes V$ -cyclic subspace generated by $v_1 \otimes v_2$. It follows from (i) and (ii) that

$$(I + U(\xi))v_1 \otimes (I + V(\xi))v_2 \in \mathcal{C},$$

$$(dI + U(\xi))v_1 \otimes (eI + V(\xi))v_2 \in \mathcal{C}.$$

Hence $U(\xi)v_1 \otimes v_2 + v_1 \otimes V(\xi)v_2 \in \mathcal{C}$ and $dU(\xi)v_1 \otimes v_2 + ev_1 \otimes V(\xi)v_2 \in \mathcal{C}$. This implies, in particular that $U(\xi)v_1 \otimes v_2 \in \mathcal{C}$. In a similar way, $U(-\xi)v_1 \otimes v_2 \in \mathcal{C}$.

Thus, $(U(\xi) \otimes I)\mathcal{C} = \mathcal{C}$. In a similar way, we deduce from (iii) and (iv) that $(U(\eta) \otimes I)\mathcal{C} = \mathcal{C}$. Hence $(U(n\xi + m\eta) \otimes I)\mathcal{C} = \mathcal{C}$ for all $n, m \in \mathbb{Z}$. Since η and ξ are rationally independent, \mathcal{C} is invariant under the unitary representation $U \otimes I$. It follows that $\mathcal{C} = \mathcal{H}_1 \otimes \mathcal{H}_2$. \square

Let $K, H, v, \mathcal{K}, \xi_1, \xi_2$ be as in the previous section. We will assume that $\xi_2 > \xi_1$ and put $\xi_3 := \xi_2 - \xi_1$. Fix a partition

$$\mathbb{N} = \bigsqcup_{i=1}^3 \bigsqcup_{a \in \mathcal{K}} \mathcal{M}_{a,i} \sqcup \mathcal{N}_a$$

of \mathbb{N} into infinite subsets. As in the previous section, to construct the desired realization we define inductively a sequence $(C_n, h_n, \alpha_n)_{n=1}^\infty$, where C_n a finite subset of \mathbb{R} , h_n is a positive real and $\alpha_n : C_n \rightarrow \mathbb{R}$ a mapping. Suppose we have already constructed this sequence up to index n . Consider two cases.

Case 1. If $n+1 \in \mathcal{N}_a$ for some $a \in \mathcal{K}$ then we denote by m_a the least positive period of a under v . Now we set

$$\begin{aligned} z_{n+1} &:= m_a n h_n, & r_n &:= n^3 m_a, \\ C_{n+1} &:= h_n \cdot \{0, 1, \dots, r_n - 1\}, \\ h_{n+1} &:= r_n h_n, \end{aligned}$$

Let $\alpha_{n+1} : C_{n+1} \rightarrow K$ be any map satisfying the following conditions

- (A1) $\alpha_{n+1}(c + z_{n+1}) = v \circ \alpha_{n+1}(c)$ for all $c \in C_{n+1} \cap (C_{n+1} - z_{n+1})$,
- (A2) for each $0 \leq i < m_a$ there is a subset $C_{n+1,i} \subset C_{n+1}$ such that

$$\begin{aligned} C_{n+1,i} - h_n &\subset C_{n+1}, \\ \alpha_{n+1}(c) &= \alpha_{n+1}(c - h_n) + v^i(a) \text{ for all } c \in C_{n+1,i} \text{ and} \\ \left| \frac{\#C_{n+1,i}}{\#C_{n+1}} - \frac{1}{m_a} \right| &< \frac{2}{nm_a}. \end{aligned}$$

Case 2. If $n+1 \in \mathcal{M}_{a,i}$ for some $a \in \mathcal{K}$ and $i = 1, 2, 3$ then we denote by m_a the least positive period of a under v . Now we set

$$\begin{aligned} z_{n+1} &:= m_a n (2h_n + \xi_i), \\ D_{n+1}^1 &:= h_n \cdot \{0, 1, \dots, m_a n - 1\}, \\ D_{n+1}^2 &:= \{j(h_n + \xi_i) + m_n n h_n \mid 0 \leq j < m_a n\}, \\ C_{n+1} &:= \bigsqcup_{j=0}^{n^2-1} (j z_{n+1} + (D_{n+1}^1 \sqcup D_{n+1}^2)), \\ h_{n+1} &:= m_a n^3 (2h_n + \xi_i), \end{aligned}$$

Let $\alpha_{n+1} : C_{n+1} \rightarrow K$ be any map satisfying the following conditions

- (B1) $\alpha_{n+1}(c + z_{n+1}) = v \circ \alpha_{n+1}(c)$ for each $c \in C_{n+1}^1 \cap (C_{n+1}^1 - z_{n+1})$,

(B2) for each $0 \leq l < m_a$ there is a subset $D_{n+1,l} \subset D_{n+1}^1$ such that

$$\begin{aligned} D_{n+1,l} - h_n &\subset D_{n+1}^1, \\ \alpha_{n+1}(c) &= \alpha_{n+1}(c - h_n) + v^l(a) \text{ for all } c \in D_{n+1,l} \text{ and} \\ \left| \frac{\#D_{n+1,l}}{\#D_{n+1}^1} - \frac{1}{m_a} \right| &< \frac{2}{nm_a}, \end{aligned}$$

(B3) $\alpha_{n+1}(c) = 1_K$ for each $c \in D_{n+1}^2$.

Thus, $C_{n+1}, h_{n+1}, \alpha_{n+1}$ are completely defined.

Denote by (X, μ, T) the (C, F) -flow associated with the sequence $(C_{n+1}, F_n)_{n \geq 0}$, where $F_n := [0, h_n]$. Let \mathcal{R} stand for the tail equivalence relation (or, equivalently, the T -orbit equivalence relation) on X . Denote by $\alpha : \mathcal{R} \rightarrow K$ the cocycle of \mathcal{R} associated with the sequence $(\alpha_n)_{n \geq 0}$. We denote by $T^{\alpha, H}$ the following flow on the space $(X \times K/H, \lambda_{K/H})$:

$$T_t^{\alpha, H}(x, k + H) := (T_t x, \alpha(T_t x, x) + k + H), \quad t \in \mathbb{R}.$$

The following lemma is an analogue of Lemma 4.2. It can be proved in a similar way by using (A2), (B2) and (B3). We leave details to the reader.

Lemma 5.3. *Let $a \in \mathcal{K}$. Then for each $\chi \in \widehat{K}$ and $j > 0$*

- (i) $U_{T^{\alpha, \chi}}(h_n) \rightarrow l_{\chi}(a) \cdot I$ as $\mathcal{N}_a - 1 \ni n \rightarrow \infty$ and
- (ii) $U_{T^{\alpha, \chi}}(jh_n) \rightarrow 0.5(l_{\chi}(ja)I + U_{T^{\alpha, \chi}}(-j\xi_i))$ as $\mathcal{M}_{a,i} - 1 \ni n \rightarrow \infty$.

Our purpose in this section is to prove the following theorem.

Theorem 5.4. *The transformation $T \times T^{\alpha, H}$ is weakly mixing and $\mathcal{M}(T \times T^{\alpha, H}) = E \cup \{2\}$.*

Proof. To show that $\mathcal{M}(T \times T^{\alpha, H}) = E \cup \{2\}$ we consider a natural decomposition of $U_{T \times T^{\alpha, H}}$ into an orthogonal sum

$$U_{T \times T^{\alpha, H}} = \bigoplus_{\chi \in \widehat{K/H}} (U_T \otimes U_{T, \chi}).$$

It is enough to prove the following:

- (a) $U_T \otimes U_T$ has a homogeneous spectrum 2 in the orthocomplement to the constants,
- (b) $U_T \otimes U_{T^{\alpha, \chi}}$ has a simple spectrum if $\chi \neq 0$,
- (c) $U_T \otimes U_{T^{\alpha, \chi}}$ and $U_T \otimes U_{T^{\alpha, \xi}}$ are unitarily equivalent if χ and ξ belong to the same \widehat{v} -orbit,
- (d) the measures of maximal spectral type of $U_T \otimes U_{T^{\alpha, \chi}}$ and $U_T \otimes U_{T^{\alpha, \xi}}$ are mutually singular if χ and ξ are not on the same \widehat{v} -orbit.

It follows from Lemma 5.3(ii) that $\text{WC}(U_T)$ contains operators $0.5(I + U_T(-\xi_1))$, $0.5(I + U_T(-2\xi_1))$, $0.5(I + U_T(-\xi_2))$ and $0.5(I + U_T(\xi_1 - \xi_2))$. Therefore we deduce (a) from Lemma 5.1.

Fix a nontrivial $\chi \in \widehat{K}$. The unitary representation $U_{T^\alpha, \chi}$ has a simple spectrum (see the proof of Theorem 3.1). Moreover,

$$\begin{aligned} U_{T^\alpha, \chi}(h_n) &\rightarrow 0.5(I + U_{T^\alpha, \chi}(-\xi_1)), & U_T(h_n) &\rightarrow 0.5(I + U_T(-\xi_1)) \\ &\text{as } \mathcal{M}_{0,1} - 1 \ni n \rightarrow \infty & \text{and} \\ U_{T^\alpha, \chi}(h_n) &\rightarrow 0.5(I + U_{T^\alpha, \chi}(-\xi_2)), & U_T(h_n) &\rightarrow 0.5(I + U_T(-\xi_2)) \\ &\text{as } \mathcal{M}_{0,2} - 1 \ni n \rightarrow \infty. \end{aligned}$$

by Lemma 5.3(ii). Since χ is nontrivial, it follows from Algebraic Lemma 1.1 that there is $a \in \mathcal{K}$ with $l_\chi(a) \neq 1$. Again by Lemma 5.3(ii),

$$\begin{aligned} U_{T^\alpha, \chi}(h_n) &\rightarrow 0.5(l_\chi(a)I + U_{T^\alpha, \chi}(-\xi_1)), & U_T(h_n) &\rightarrow 0.5(I + U_T(-\xi_1)) \\ &\text{as } \mathcal{M}_{a,1} - 1 \ni n \rightarrow \infty & \text{and} \\ U_{T^\alpha, \chi}(h_n) &\rightarrow 0.5(l_\chi(a)I + U_{T^\alpha, \chi}(-\xi_2)), & U_T(h_n) &\rightarrow 0.5(I + U_T(-\xi_2)) \\ &\text{as } \mathcal{M}_{a,2} - 1 \ni n \rightarrow \infty. \end{aligned}$$

Therefore Lemma 5.2 implies (b).

As in the proof of Theorem 4.3 we can define a transformation $S_{\bar{z}}$ of (X, μ) by the formula (3-1). Then $S_{\bar{z}} \in C(T)$. It follows from (A1), (B1) and the definition of C_{n+1} that (3-2) is satisfied. Hence by Lemma 3.3, the cocycle $\alpha \circ S_{\bar{z}}$ is cohomologous to $v \circ \alpha$. Therefore the unitary representations $U_{T, \chi}$ and $U_{T, \xi}$ are unitarily equivalent whenever χ and ξ lie on the same orbit of \widehat{v} . This yields (c).

To prove (d), we first find $a \in \mathcal{G}$ such that $l_\chi(a) \neq l_\xi(a)$ (see claim (ii) of Algebraic Lemma). It follows from Lemma 5.3(i) that

$$U_T(h_n) \otimes U_{T^\alpha, \chi}(h_n) \rightarrow l_\chi(a)I \quad \text{and} \quad U_T(h_n) \otimes U_{T^\alpha, \xi}(h_n) \rightarrow l_\xi(a)I$$

as $\mathcal{N}_a - 1 \ni n \rightarrow \infty$. This implies (d).

Finally, since $\mathcal{M}(T \times T^{\alpha, H}) \not\cong 1$, it follows that $T \times T^{\alpha, H}$ is weakly mixing. \square

Remark 5.5. It follows from Lemma 5.3(ii) that $U_{T^\alpha, H}(h_n) \rightarrow 0.5(I + U_{T^\alpha, H}(-\xi_i))$ as $\mathcal{M}_{0,i} - 1 \ni n \rightarrow \infty$, $j = 1, 2, 3$. As in Proposition 4.3 we can deduce from this fact that $\mathcal{M}(T^{\alpha, H}) = \mathcal{M}(T_t^{\alpha, H})$ for each $t \neq 0$.

6. SPECTRAL MULTIPLICITIES FOR ERGODIC ACTIONS OF OTHER GROUPS

The main result of the paper extends partly to actions of some other locally compact second countable Abelian groups G . If G is compact then each ergodic action T of G has a pure point spectrum and $\mathcal{M}(T) = \{1\}$. Therefore from now on we assume that G is non-compact.

Corollary 6.1. *Let G be a torsion free discrete countable Abelian group and let E be a subset of \mathbb{N} such that $E \cap \{1, 2\} \neq \emptyset$. Then there is a weakly mixing free action S of G such that $\mathcal{M}(S) = E$.*

Proof. In the case when $G = \mathbb{Z}$ see [Da3] and references therein. Consider now the case when $G \neq \mathbb{Z}$. Then there is an embedding ϕ of G into \mathbb{R} such that the subgroup $\phi(\mathbb{R})$ is dense in \mathbb{R} . Indeed, it is well known that G embeds into $\mathbb{Q}^\mathbb{N}$. In turn, the later group obviously embeds into \mathbb{R} . It remains to note that if an infinite subgroup of \mathbb{R} is not isomorphic to \mathbb{Z} then it is dense in \mathbb{R} .

By Theorem 0.1, there is a weakly mixing action T of \mathbb{R} such that $\mathcal{M}(T) = E$. Then the composition $T \circ \phi = (T_{\phi(g)})_{g \in G}$ is a weakly mixing action of G with $\mathcal{M}(T \circ \phi) = \mathcal{M}(T) = E$. \square

The first claim of the following lemma is, in fact, a slight generalization of Theorem 4.1. If we replace (relax) “weak mixing” in its statement with “ergodic” then it follows from Theorem 4.1 via Proposition 1.1.

Lemma 6.2. *Let A be a compact second countable Abelian group. Let E be a subset of \mathbb{N} with $1 \in E$.*

- (i) *There is a weakly mixing free action W of $\mathbb{R} \times A$ such that $\mathcal{M}(W) = E$.*
- (ii) *For each torsion free discrete countable Abelian group G , there is a weakly mixing free action W of $G \times A$ such that $\mathcal{M}(W) = E$.*

Proof. (i) Let the objects K, H, v be defined exactly as in Section 4. We now set $K' := K \times A$, $H' := H \times \{0\}$ and $v' := v \times \text{Id}$. It is straightforward that

$$(6-1) \quad L(\widehat{K'}, \widehat{K'/H'}, \widehat{v'}) = L(\widehat{K}, \widehat{K/H}, \widehat{v}) = E.$$

Moreover, the subgroup of v' -periodic points is countable and dense in K' . We now construct the skew product flow $T^{\alpha', H'}$ in the same way as in Section 4 but with K', H', v' instead of K, H, v . We note that $T^{\alpha', H'}$ acts on the space $(Y, \nu) := (X \times K/H \times A, \mu \times \lambda_{K/H} \times \lambda_A)$. Denote by $W = (W_{t,a})_{(t,a) \in \mathbb{R} \times A}$ the action of the product group $\mathbb{R} \times A$ on (Y, ν) generated by $T^{\alpha', H'}$ and the action of A by rotations along the third coordinate. Then W is free. Since $T^{\alpha', H'}$ is weakly mixing, so is W . Denote by U_W the corresponding Koopman unitary representation of $\mathbb{R} \times A$ in $L^2_0(Y, \nu)$. We have a decomposition

$$L^2(Y, \nu) = \bigoplus_{\chi \in \widehat{K/H}, \eta \in \widehat{A}} \mathcal{H}_{\chi, \eta},$$

where $\mathcal{H}_{\chi, \eta} := L^2(X, \mu) \otimes \chi \otimes \eta$. We know from Section 4 that $\mathcal{H}_{\chi, \eta}$ is a $U_{T^{\alpha', H'}}$ -cyclic subspace for each pair χ, η . It is also a U_W -cyclic subspace. The unitary operator $U_W(0, a)$ acts on $\mathcal{H}_{\chi, \eta}$ by multiplying on $\eta(a)$. Hence if $\sigma_{\chi, \eta}$ is a measure of maximal spectral type of $U_{T^{\alpha', H'}} \upharpoonright \mathcal{H}_{\chi, \eta}$ then the measure $\sigma_{\chi, \eta} \times \delta_\eta$ on $\widehat{R} \times \widehat{A}$ is a measure of maximal spectral type of $U_W \upharpoonright \mathcal{H}_{\chi, \eta}$. As was shown in Section 4, if (χ, η) and (χ', η') belong to different v' -orbits then $\sigma_{\chi, \eta} \perp \sigma_{\chi', \eta'}$. It follows that $\sigma_{\chi, \eta} \times \delta_\eta \perp \sigma_{\chi', \eta'} \times \delta_{\eta'}$. On the other hand, if (χ, η) and (χ', η') belong to the same v' -orbit then $\sigma_{\chi, \eta} \sim \sigma_{\chi', \eta'}$. Moreover, $\eta = \eta'$ by the definition of v' . Hence $\sigma_{\chi, \eta} \times \delta_\eta \sim \sigma_{\chi', \eta'} \times \delta_{\eta'}$. These facts plus (6-1) imply that $\mathcal{M}(W) = E$.

(ii) Consider two cases. If G is not \mathbb{Z} then (ii) follows from (i) in the very same way as Corollary 6.1 follows from Theorem 0.1. If G is \mathbb{Z} then we need to modify the proof of the main result from [Da3] (only the case when $E \ni 1$) in the very same way as we modified the proof of Theorem 4.1 in (i). \square

Let T_1 and T_2 be probability preserving ergodic actions of locally compact second countable Abelian groups G_1 and G_2 respectively. Let $T_1 \otimes T_2$ stand for the product action $(g_1, g_2) \mapsto T_1(g_1) \times T_2(g_2)$ of the product group $G_1 \times G_2$. It is easy to see that

- (•) if T_1 has a simple spectrum then $\mathcal{M}(T_1 \otimes T_2) = \mathcal{M}(T_2) \cup \{1\}$.

As far as we know, this fact was first used in [Fi] for \mathbb{Z}^2 -actions.

Corollary 6.3. *Let $E \ni 1$. If one of the following conditions is satisfied*

- (i) *G contains a closed one-parameter subgroup,*
- (ii) *$G = D \times F$, where D is a torsion free discrete countable Abelian group and F is a locally compact second countable Abelian group then*

there is a free weakly mixing action T of G such that $\mathcal{M}(T) = E$.

Proof. (i) It follows from [HR, Theorem 24.30] that G is topologically isomorphic to a product $\mathbb{R} \times G'$, where G' is a locally compact Abelian group.

Suppose first that G' is non-compact. We now claim that there is a weakly mixing free G' -action with a simple spectrum. To prove this claim we need several standard auxiliary facts which we state here without proof.

- Let \mathcal{A} be the set of all G' -actions on a standard probability space (X, μ) . A G' -action is considered as a continuous map from G' to the Polish (in the weak topology) group of all transformations of (X, μ) . Then \mathcal{A} endowed with the topology of uniform convergence on the compact subsets in G' is Polish.
- The conjugacy class of every free G' -action is dense in \mathcal{A} .
- The subset of all weakly mixing G' -actions and the subset of all G' -actions with a simple spectrum are both G_δ in \mathcal{A} .
- There is a weakly mixing free G' -action and there is a free G' -action with a simple spectrum.

The claim follows from them via a *generic argument*. Now we deduce the assertion of the Corollary 6.3 from Theorem 0.1 and (\bullet) .

Consider now the second case when G' is compact. Then the assertion of the Corollary 6.3 follows from Lemma 6.2(i).

(ii) is proved in a similar way by replacing the references to Theorem 0.1 and Lemma 6.2(i) with references to Corollary 6.1 and Lemma 6.2(ii) respectively. \square

We note that if G is connected then (i) is satisfied. If G has no non-trivial compact subgroups then one of the conditions of Corollary 6.3 is satisfied.

We claim that Theorem 0.1, the case $2 \in E$, holds true if we replace \mathbb{R} -actions with actions of groups G which are isomorphic to the product of \mathbb{R}^d with torsion free discrete Abelian groups. However to prove this fact one has to pass all the way of Section 5 by adjusting all the arguments from there to the case of \mathbb{R}^d -actions. We leave this routine to the reader.

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INSTITUTE FOR LOW TEMPERATURE PHYSICS & ENGINEERING OF NATIONAL ACADEMY OF SCIENCES OF UKRAINE, 47 LENIN AVE., KHARKOV, 61164, UKRAINE
E-mail address: alexandre.danilenko@gmail.com

FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, NICOLAUS COPERNICUS UNIVERSITY, UL. CHOPINA 12/18, 87-100 TORUŃ, POLAND

AND INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES UL. ŚNIADECKICH 8, 00-950 WARSAW, POLAND
E-mail address: mlem@mat.uni.torun.pl